Texas A&M University, August 19-22, 2013

International Workshop on Nuclear Dynamics and Thermodynamics

The Mott Effect

Gerd Röpke, Rostock

Outline

- Quantum statistics of many-particle systems
- Equation of state of warm nuclear matter (subsaturation density)
- Symmetry energy
- Conclusions

Dense plasmas and Mott effect

- Conductivity in Coulomb systems (charged particle systems): doped semiconductors, mercury vapor, hydrogen plasma,…
- At increasing density: Screening of the Coulomb interaction, Debye potential in a simple approximation

$$\frac{e^2}{r} \to \frac{e^2}{r} \mathrm{e}^{-r/r_{\mathrm{Debye}}} \qquad r_{\mathrm{Debye}} = \left(\frac{T}{4\pi e^2 n_e} \right)^{1/2}$$

- Mott: Metal-insulator phase transition (MIT).

  Above a critical density, bound charge carriers become itinerant
- Bound states disappear, electrons are conducting, metallic state
- Many-particle theory: Dynamical screening, dynamical self-energy; quantum statistical approach: in-medium atomic wave equation

W. Kraeft, D. Kremp, W. Ebeling, G. Roepke (1986): *Quantum statistics of charged particle systems.*

Bound states in dense (fermion) systems

Many-particle system	Coulomb plasma
constituents	electrons, ions
Bound states	H
(atoms,molecules)	H^-, H_2^+
	H_2
Increasing density:	
Condensed phase	metal (conducting)
Equation of state,	
Phase transition	plasma phase trans.
Microscopic process	screening

Bound states disappear at high densities.

Bound states in dense (fermion) systems

Many-particle system	Coulomb plasma	nuclear system
constituents	electrons, ions	neutrons, protons
Bound states	H	d
(atoms,molecules)	H^-, H_2^+	t, ^{3}He (h)
	H_2	^{4}He (α)
Increasing density:		
Condensed phase	metal (conducting)	nuclear matter
Equation of state,		
Phase transition	plasma phase trans.	liquid-gas phase trans.
Microscopic process	screening	Pauli blocking

Bound states disappear at high densities.

Work out the QS theory for nuclear systems, medium modifications

Quantum statistics of hot nuclear systems

Nuclear Physics **A379** (1982) 536–552

PARTICLE CLUSTERING AND MOTT TRANSITIONS IN NUCLEAR MATTER AT FINITE TEMPERATURE (I). Method and general aspects

(II): G. R., M. Schmidt, L. Muenchow, H. Schulz, NPA **399** (1983), 587.

G. RÖPKE

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L. MÜNCHOW

Zentralinstitut für Kernforschung, Rossendorf, GDR

and

H. SCHULZ*

The Niels Bohr Institute, University of Copenhagen, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

Thermodynamic (Matsubara) Green's function approach, Beth-Uhlenbeck Eq.
The denotation Mott effect was invented in analogy to plasma physics. But:
The microscopic process is Pauli blocking instead of Coulomb screening.

Some consequences

- Clustering (correlations dominate) at low densities:

  Nuclear statistical equilibrium (NSE), chemical picture

  Virial expansion, Beth-Uhlenbeck formula
- Higher densities: medium effects

  Mean-field approximation: Skyrme, RMF
- Bound states as quasiparticles, quasiparticle shift
- In-medium Schroedinger equation, self-energy and Pauli blocking, Bound states disappear (Mott effect), generalized Beth-Uhlenbeck formula, composition, in-medium cross sections

  Physical picture: correlations and spectral functions
- Phase transition, quantum condensates, quartetting.

Effective wave equation for the deuteron in matter

In-medium two-particle wave equation in mean-field approximation

$$\left(\frac{p_1^2}{2m_1}+\Delta_1+\frac{p_2^2}{2m_2}+\Delta_2\right)\Psi_{d,P}(p_1,p_2)+\sum_{p_1',p_2'}(1-f_{p_1}-f_{p_2})V(p_1,p_2;p_1',p_2')\Psi_{d,P}(p_1',p_2')$$

$$=E_{d,P}\Psi_{d,P}(p_1,p_2)$$

Add self-energy

Pauli-blocking

Thouless criterion

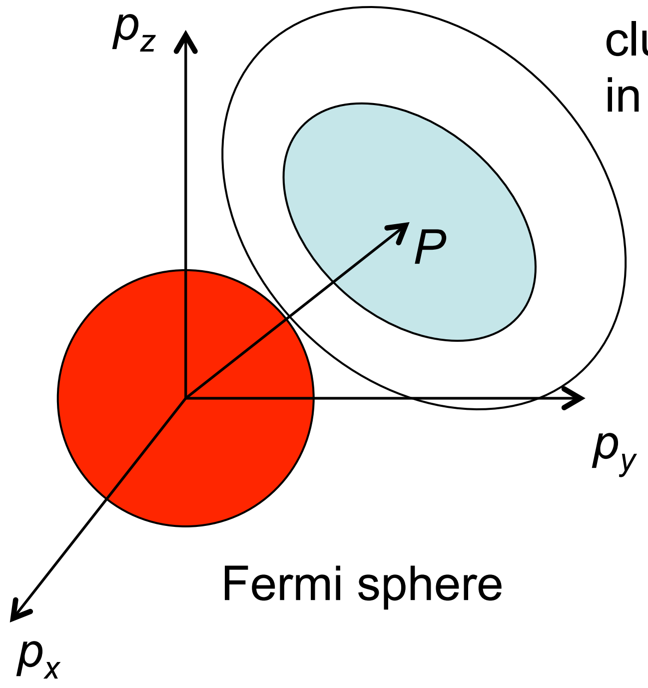
$$E_d(T,\mu)=2\mu$$

Fermi distribution function

$$f_p=\left[e^{(p^2/2m-\mu)/k_BT}+1\right]^{-1}$$

BEC-BCS crossover:
Alm et al.,1993

Pauli blocking – phase space occupation



cluster wave function (deuteron, alpha,…)
in momentum space

P - center of mass momentum

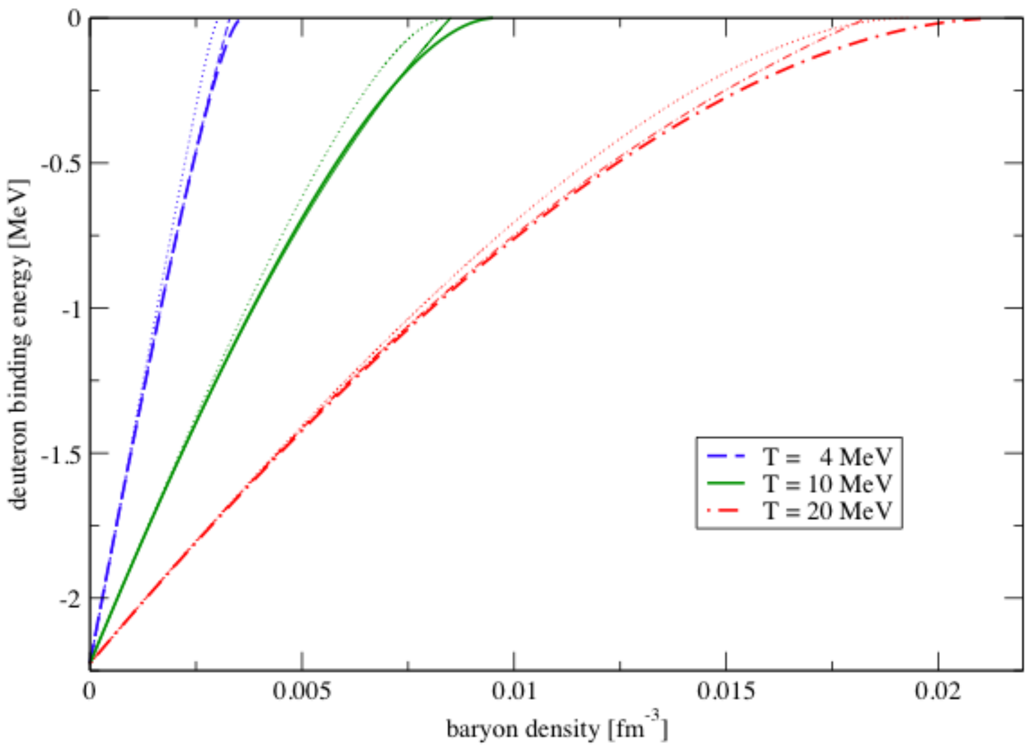
The Fermi sphere is forbidden,
deformation of the cluster wave function
in dependence on the c.o.m. momentum *P*

momentum space

The deformation is maximal at $P = 0$.
It leads to the weakening of the interaction
(disintegration of the bound state).

Shift of the deuteron bound state energy

Dependence on nucleon density, various temperatures, zero center of mass momentum



thin lines:

fit formula

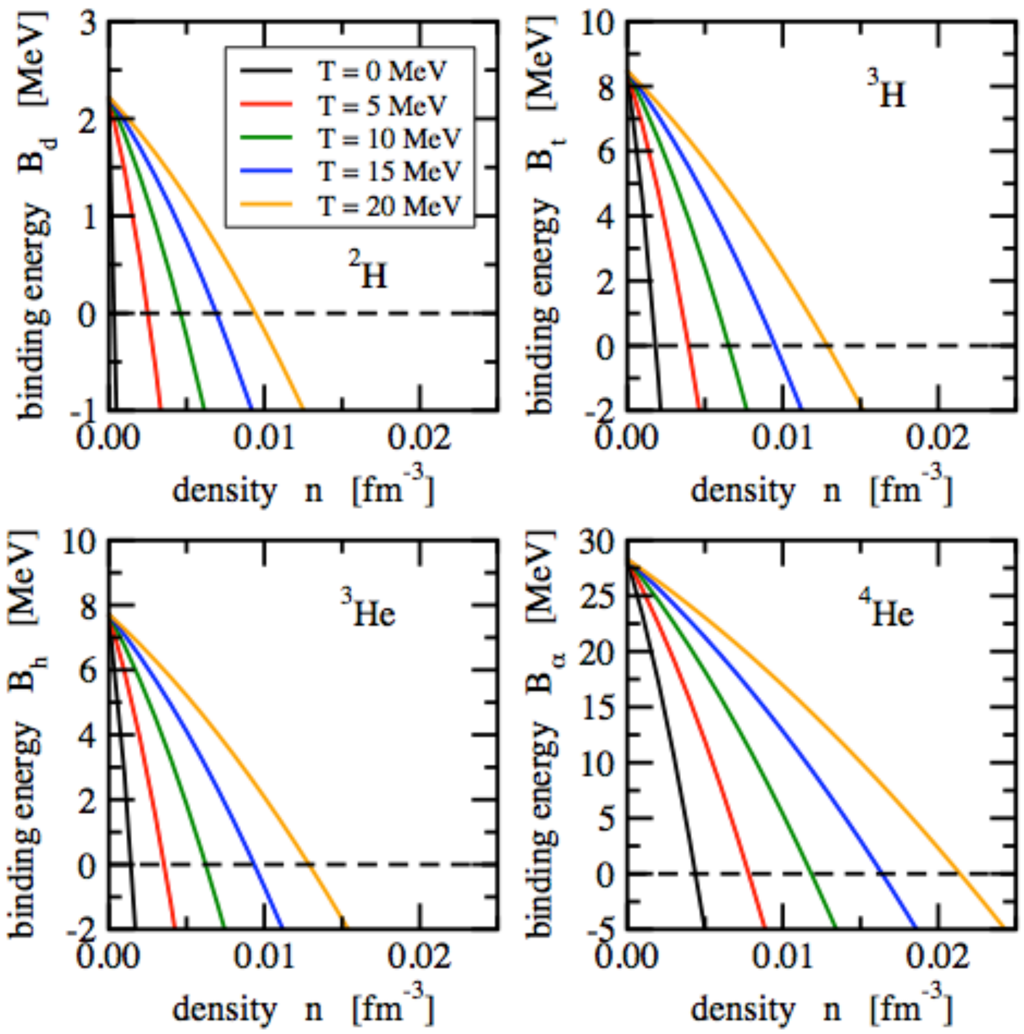
G.R., NP A **867**, 66 (2011)

Few-particle Schrödinger equation in a dense medium

4-particle Schrödinger equation with medium effects

$$
\begin{aligned}
&\left(\left[E^{HF}(p_1) + E^{HF}(p_2) + E^{HF}(p_3) + E^{HF}(p_4)\right]\right)\Psi_{n,P}(p_1,p_2,p_3,p_4) \\
&+ \sum_{p_1',p_2'} (1 - f_{p_1} - f_{p_2}) V(p_1,p_2;p_1',p_2')\Psi_{n,P}(p_1',p_2',p_3,p_4) \\
&+ \{permutations\} \\
&= E_{n,P}\Psi_{n,P}(p_1,p_2,p_3,p_4)
\end{aligned}
$$

Shift of Binding Energies of Light Clusters



Symmetric matter

G.R., PRC **79**, 014002 (2009)
S. Typel et al.,
PRC **81**, 015803 (2010)

G.R., NP A **867**, 66 (2011)

Composition of dense nuclear matter

$$n_p(T,\mu_p,\mu_n) = \frac{1}{V}\sum_{A,\nu,K} Z_A f_A\{E_{A,\nu K} - Z_A\mu_p - (A-Z_A)\mu_n\}$$
$$n_n(T,\mu_p,\mu_n) = \frac{1}{V}\sum_{A,\nu,K} (A-Z_A) f_A\{E_{A,\nu K} - Z_A\mu_p - (A-Z_A)\mu_n\}$$

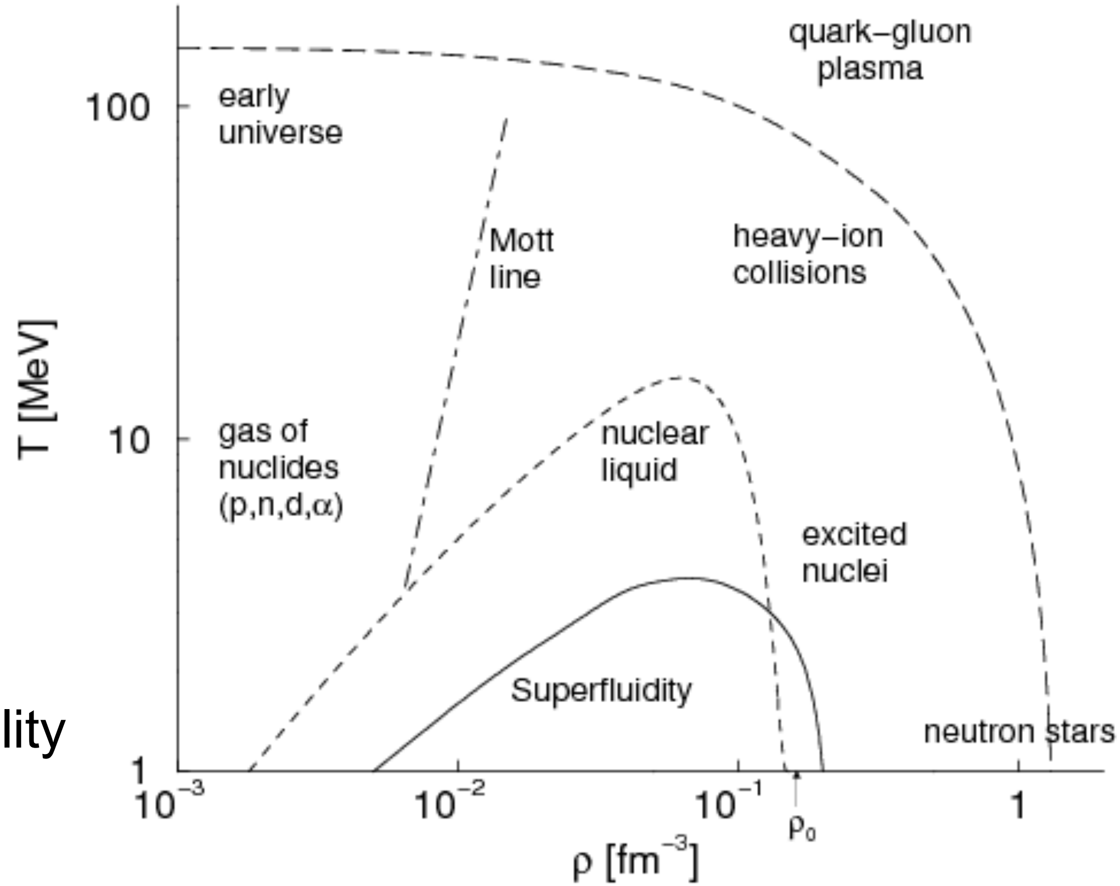
mass number A
charge Z_A
energy $E_{A,\nu,K}$
ν: internal quantum number

$$f_{A(z)} = \frac{1}{\exp(z/T) - (-1)^A}$$

- Inclusion of excited states and continuum correlations
- Medium effects:
  self-energy and Pauli blocking shifts of binding energies,
  Coulomb corrections due to screening (Wigner-Seitz,Debye)
- Bose-Einstein condensation

Symmetric nuclear matter: Phase diagram

Old figure,
see also Phil Siemens



Mott line:

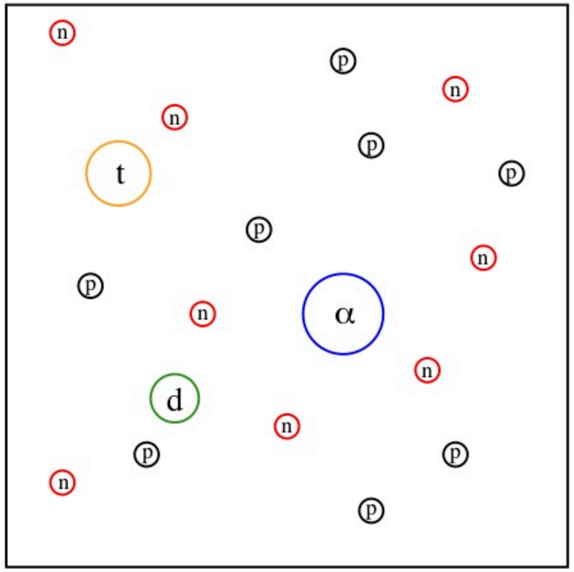
Smooth transition
no sharp changes,
bound states are
merging the continuum,
(resonances)
Levinson theorem
But thermodynamic instability
may be induced

Nuclear statistical equilibrium (NSE)

Chemical picture:

Ideal mixture of reacting components

Mass action law



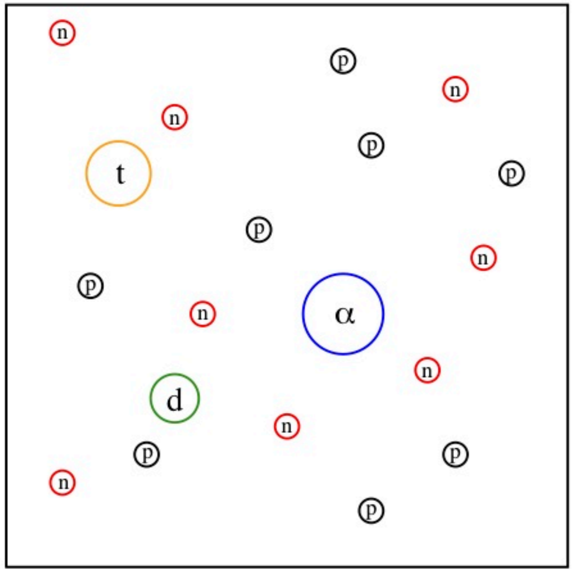
Interaction between the components
internal structure: Pauli principle

Nuclear statistical equilibrium (NSE)

Chemical picture:

Ideal mixture of reacting components

Mass action law

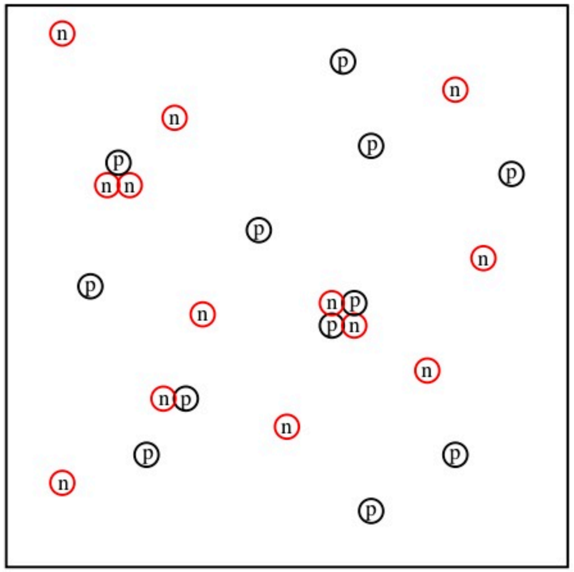


Interaction between the components
internal structure: Pauli principle

Physical picture:

"elementary" constituents

and their interaction



Quantum statistical (QS) approach,
quasiparticle concept, virial expansion

First summary

- The low density limit is well described within a chemical picture (composition, mass action law/NSE, chemical constants).
- With increasing density, medium effects are important. Near saturation, a mean-field description is possible.
- A quantum statistical approach joins both limits. Go beyond single-particle to treat correlations/bound states.
- Bound states (quasiparticles) die out due to medium effects (self-energy, Pauli blocking), Mott effect.
- A consistent description can be given calculating the spectral function for few-nucleon states (physical picture).
- Consequences: limits of the single-particle (mean-field) approaches
  – no real quantum correlations (bound states);
  limits of the NSE – no medium corrections ($n<10^{-4}$ fm^{-3}).
- Reconsider results obtained with ideal mixtures (Albergo, isoscaling,…)

Experimental verification

Heavy Ion Collisions

- Test of the nuclear matter equation of state: in-medium corrections are necessary
- Symmetry energy of nuclear matter: clustering effects at low densities must be taken into account

Nimrod @ TAMU,
40Ar + 112,124Sn,
64Zn + 112,124Sn;   47 A MeV

Yields of p, (n), d, t, ^{3}He, ^{4}He,…

Open questions: freeze-out model or dynamical transport models?
Identification of the source?
In any case, a consistent quantum statistical description is necessary.

EOS at low densities from HIC

PRL **108,** 172701 (2012) PHYSICAL REVIEW LETTERS week ending 27 APRIL 2012

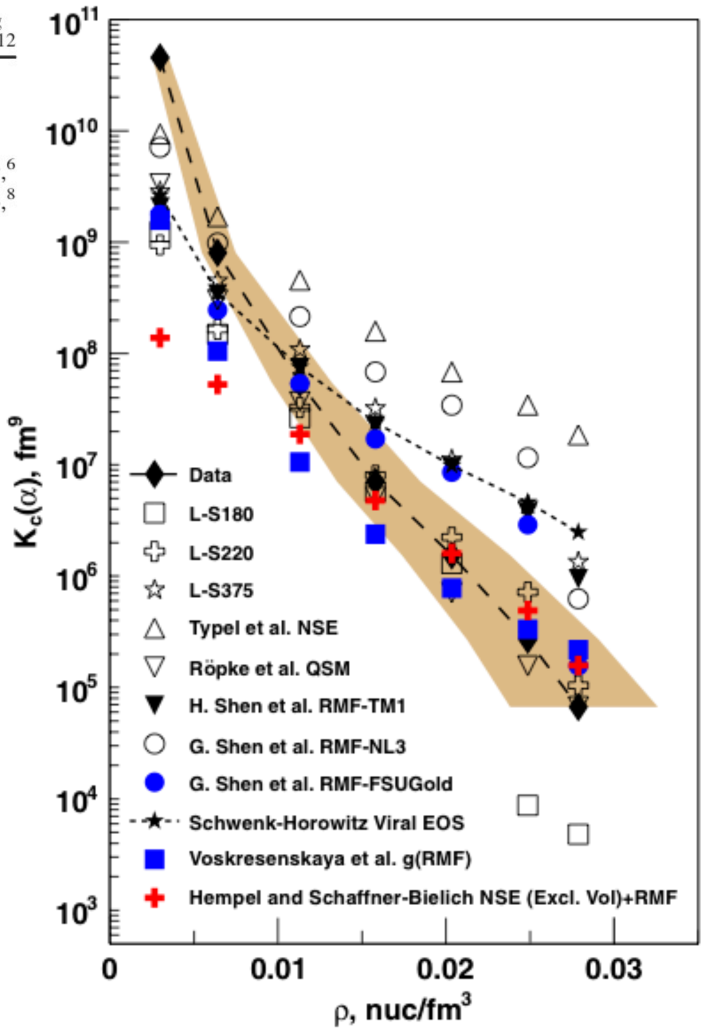
Laboratory Tests of Low Density Astrophysical Nuclear Equations of State

L. Qin,[1] K. Hagel,[1] R. Wada,[2,1] J. B. Natowitz,[1] S. Shlomo,[1] A. Bonasera,[1,3] G. Röpke,[4] S. Typel,[5] Z. Chen,[6] M. Huang,[6] J. Wang,[6] H. Zheng,[1] S. Kowalski,[7] M. Barbui,[1] M. R. D. Rodrigues,[1] K. Schmidt,[1] D. Fabris,[8] M. Lunardon,[8] S. Moretto,[8] G. Nebbia,[8] S. Pesente,[8] V. Rizzi,[8] G. Viesti,[8] M. Cinausero,[9] G. Prete,[9] T. Keutgen,[10] Y. El Masri,[10] Z. Majka,[11] and Y. G. Ma[12]

Yields of clusters from HIC: p, n, d, t, h, α

chemical constants

$$K_c(A,Z) = \rho_{(A,Z)}/[(\rho_p)^Z(\rho_n)^N]$$



Densitometers

- Albergo (NSE) cannot be used.
- Modified Albergo: use only the bound state yields, not the free nucleon yields
  [S. Kowalski et al., PRC **75**, 014601 (2007)]
- Fluctuations – see talks yesterday
- Joe Natowitz: chemical constants $K_c(A,Z)$
  Volume is needed. Mekjian coalescence model.
  Good agreement with QS calculations

$$K_\alpha = \frac{Y_\alpha}{Y_p^4}\frac{Y_h^2}{Y_t^2}(Y_p + Y_n + 2Y_d + 3Y_t + 3Y_h + 4Y_\alpha)^3$$

Low-density limit: NSE

$$\ln K_\alpha^{\mathrm{NSE}} = 3\ln n_B + \frac{E_\alpha + 2E_h - 2E_t}{T} + \frac{9}{2}\ln\left(\frac{2\pi\hbar^2}{mT}\right) - \ln 2$$

PHYSICAL REVIEW C **88**, 024609 (2013)

Density determinations in heavy ion collisions

G. Röpke,[1,*] S. Shlomo,[2] A. Bonasera,[2,3] J. B. Natowitz,[2] S. J. Yennello,[2] A. B. McIntosh,[2] J. Mabiala,[2] L. Qin,[2]
S. Kowalski,[2] K. Hagel,[2] M. Barbui,[2] K. Schmidt,[2] G. Giulani,[2] H. Zheng,[2] and S. Wuenschel[2]
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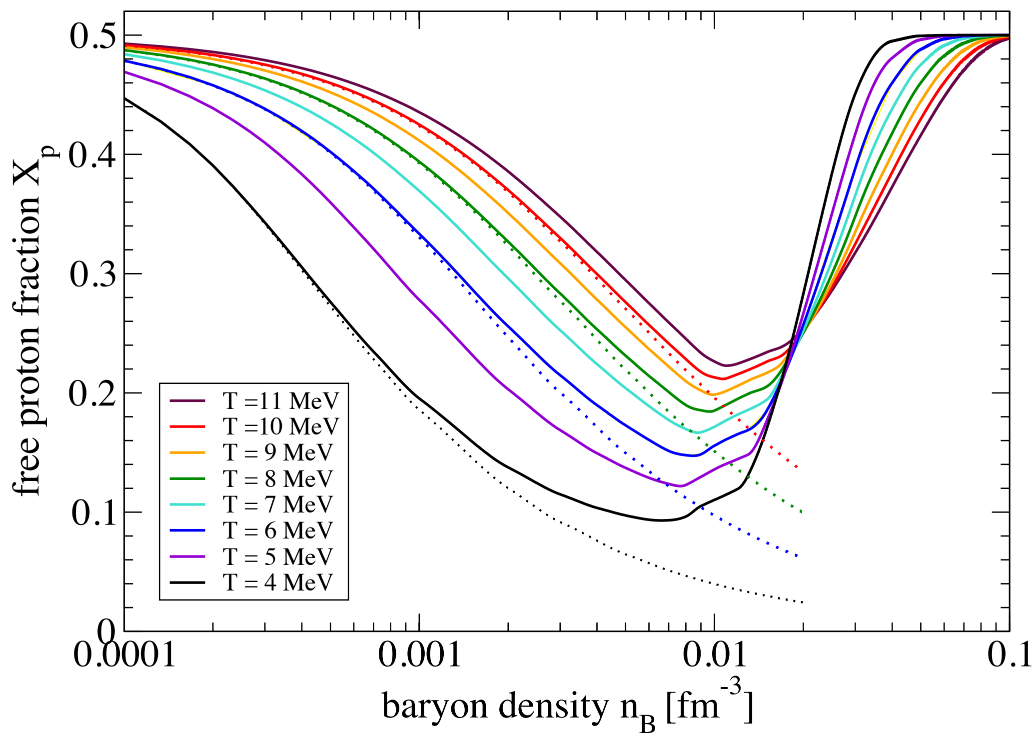
(Received 3 June 2013; revised manuscript received 15 July 2013; published 14 August 2013)

Pauli blocking in symmetric matter



Free proton fraction as function of density and temperature in symmetric matter.
QS calculations (solid lines) are compared with the NSE results (dotted lines).
Mott effect in the region $n_{saturation}/5$.

Mott points from cluster yields

PRL **108,** 062702 (2012) PHYSICAL REVIEW LETTERS week ending 10 FEBRUARY 2012

Experimental Determination of In-Medium Cluster Binding Energies and Mott Points in Nuclear Matter

K. Hagel,[1] R. Wada,[2,1] L. Qin,[1] J. B. Natowitz,[1] S. Shlomo,[1] A. Bonasera,[1,3] G. Röpke,[4] S. Typel,[5] Z. Chen,[2] M. Huang,[2] J. Wang,[2] H. Zheng,[1] S. Kowalski,[6] C. Bottosso,[1] M. Barbui,[1] M. R. D. Rodrigues,[1] K. Schmidt,[1] D. Fabris,[7] M. Lunardon,[7] S. Moretto,[7] G. Nebbia,[7] S. Pesente,[7] V. Rizzi,[7] G. Viesti,[7] M. Cinausero,[8] G. Prete,[8] T. Keutgen,[9] Y. El Masri,[9] and Z. Majka[10]

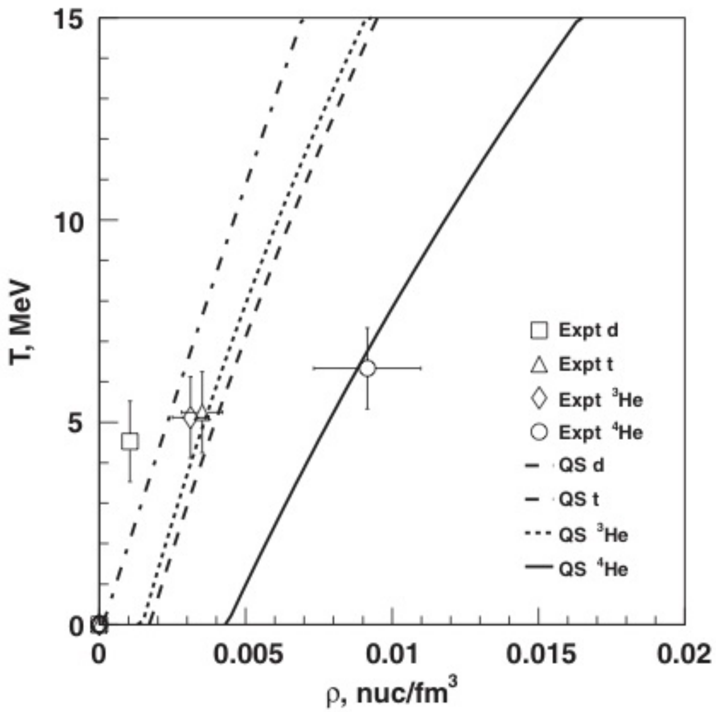
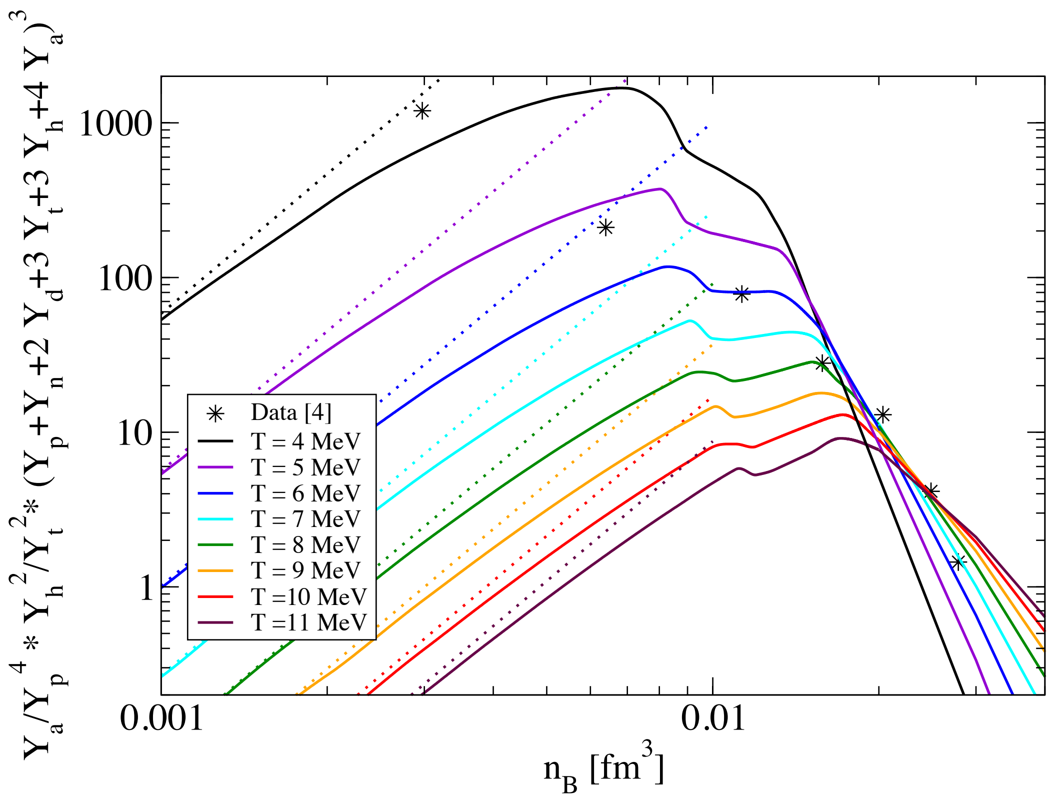


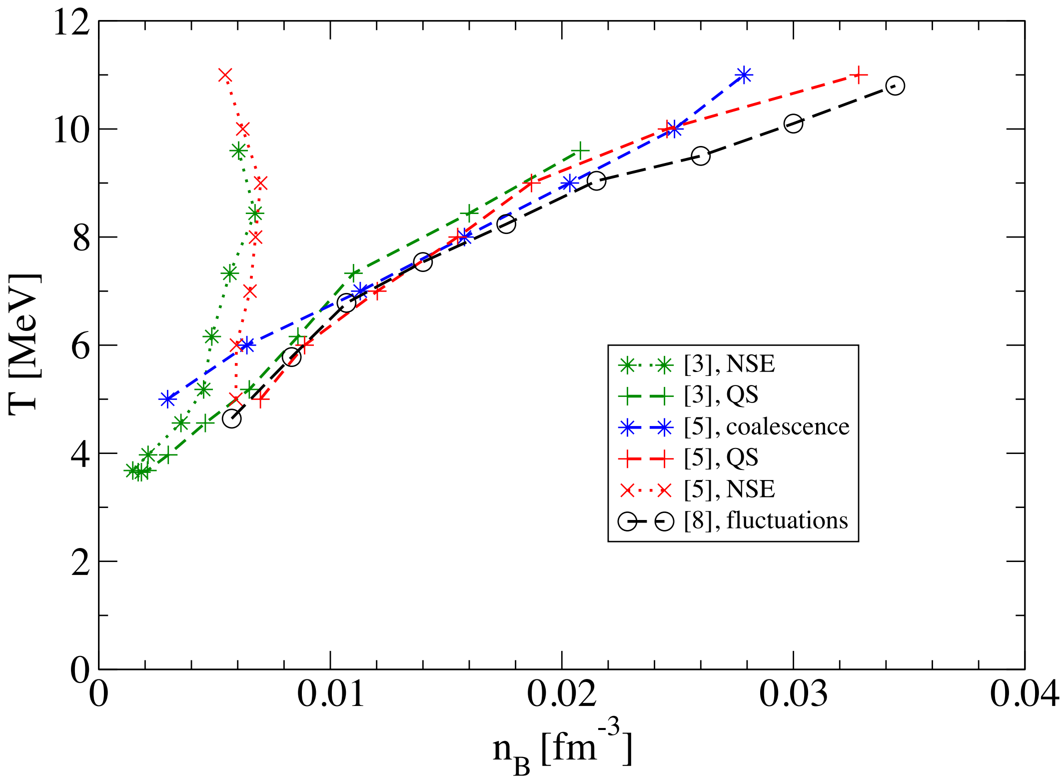
FIG. 3. Comparison of experimentally derived Mott point densities and temperatures with theoretical values. Symbols represent the experimental data. Estimated errors on the temperatures are 10% and on the densities 20%. Lines show polynomial fits to the Mott points presented in Ref. [1].

Density determination from light cluster yields



Cluster yield ratios to infer the freeze-out density:
Experimental data (stars) for T = 5; 6; 7; 8; 9; 10; 11 MeV (increasing density)
in comparison with the NSE values (thin dotted lines)
and QS calculations (bold straight lines).

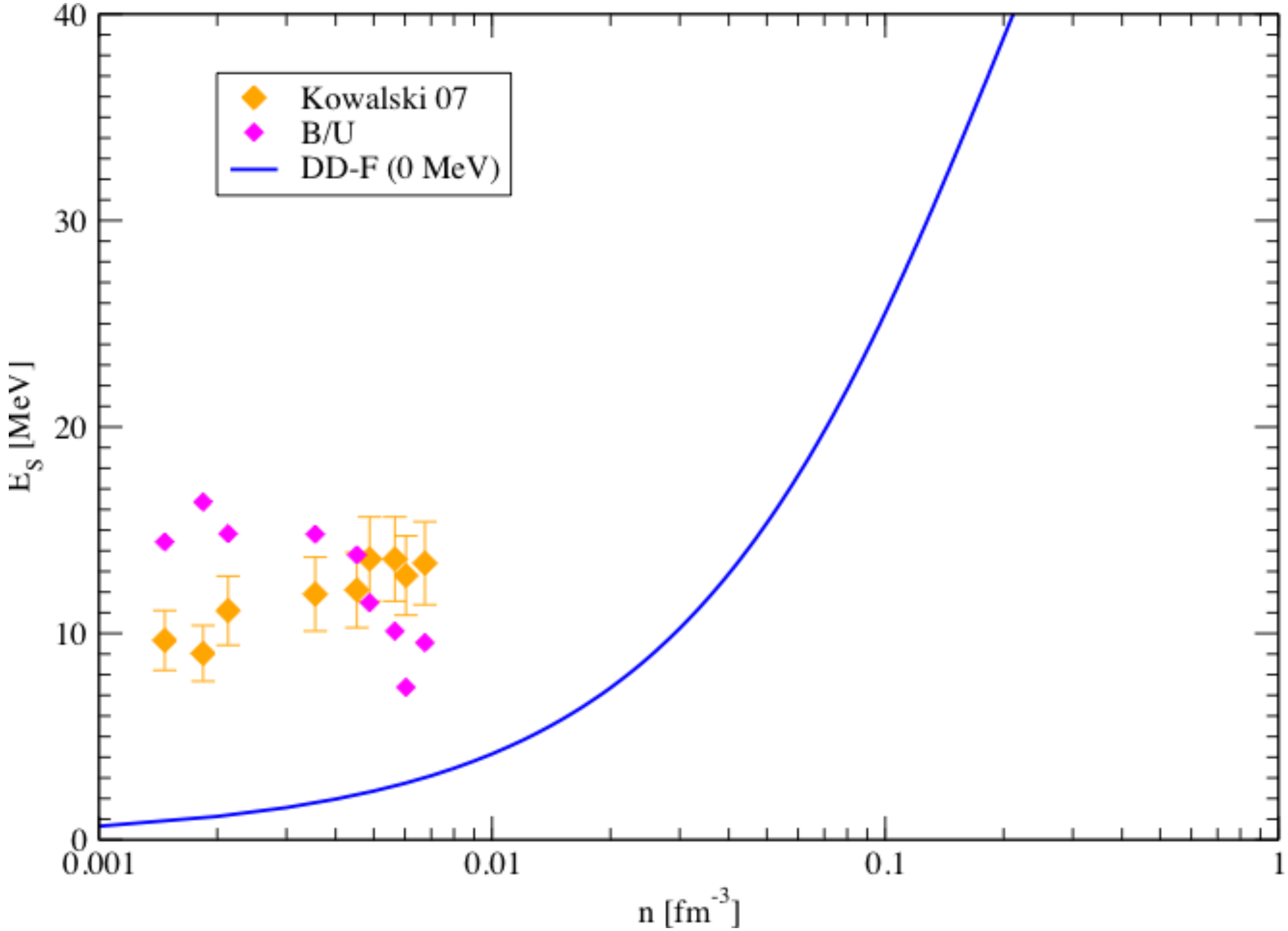
Determination of thermodynamic parameters from light cluster yields in HIC



Analysis of different measured light cluster yields ([3] – Kowalski et al., [5] – Natowitz et al., [8] – Yennello et al.) to infer the freeze-out density and temperature values.
In-medium quantum statistical (QS) agrees with coalescence and fluctuation analysis, NSE gives too low densities.

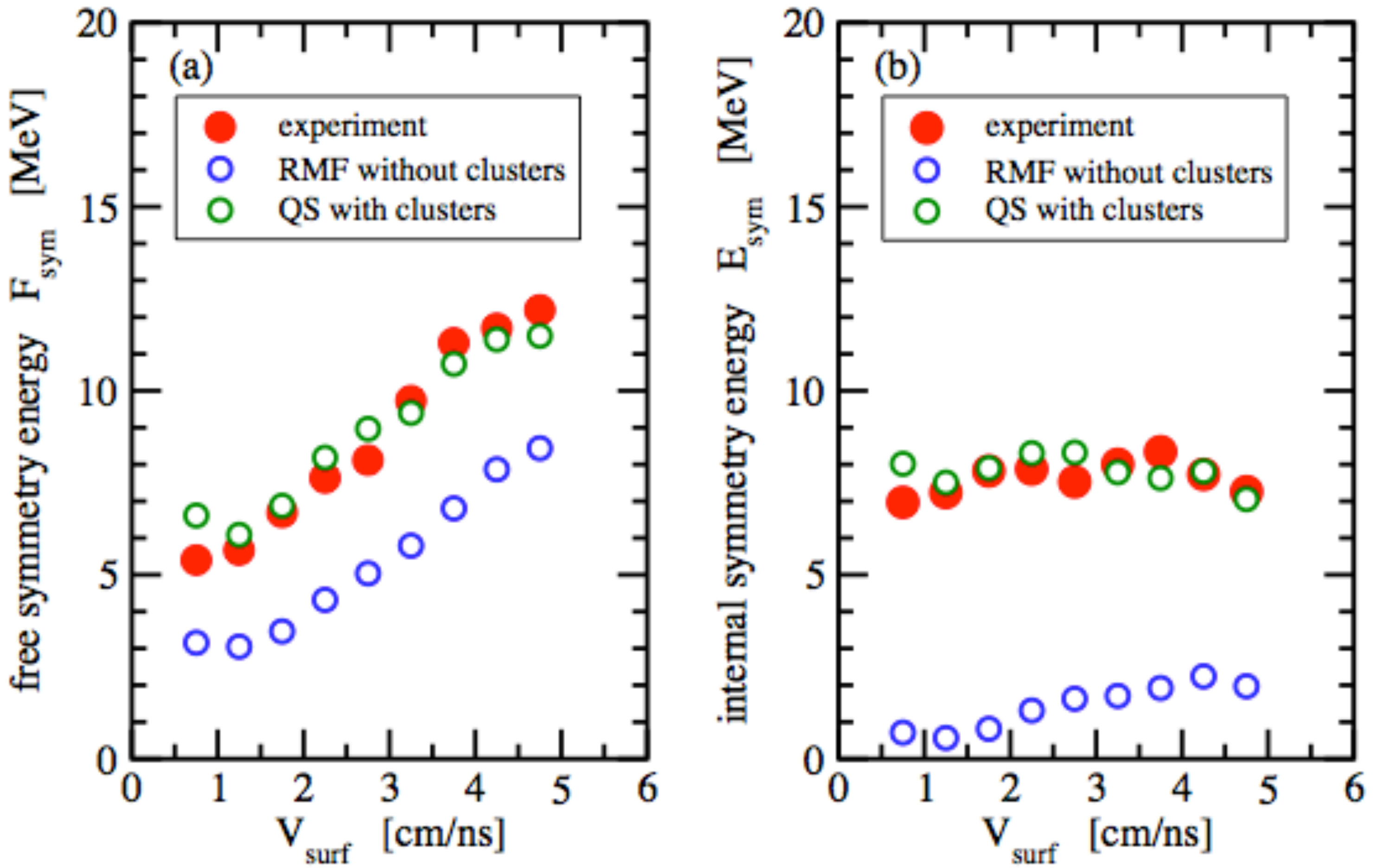
Symmetry energy

Heavy-ion collisions, spectra of emitted clusters,
temperature (3 - 10 MeV), free energy



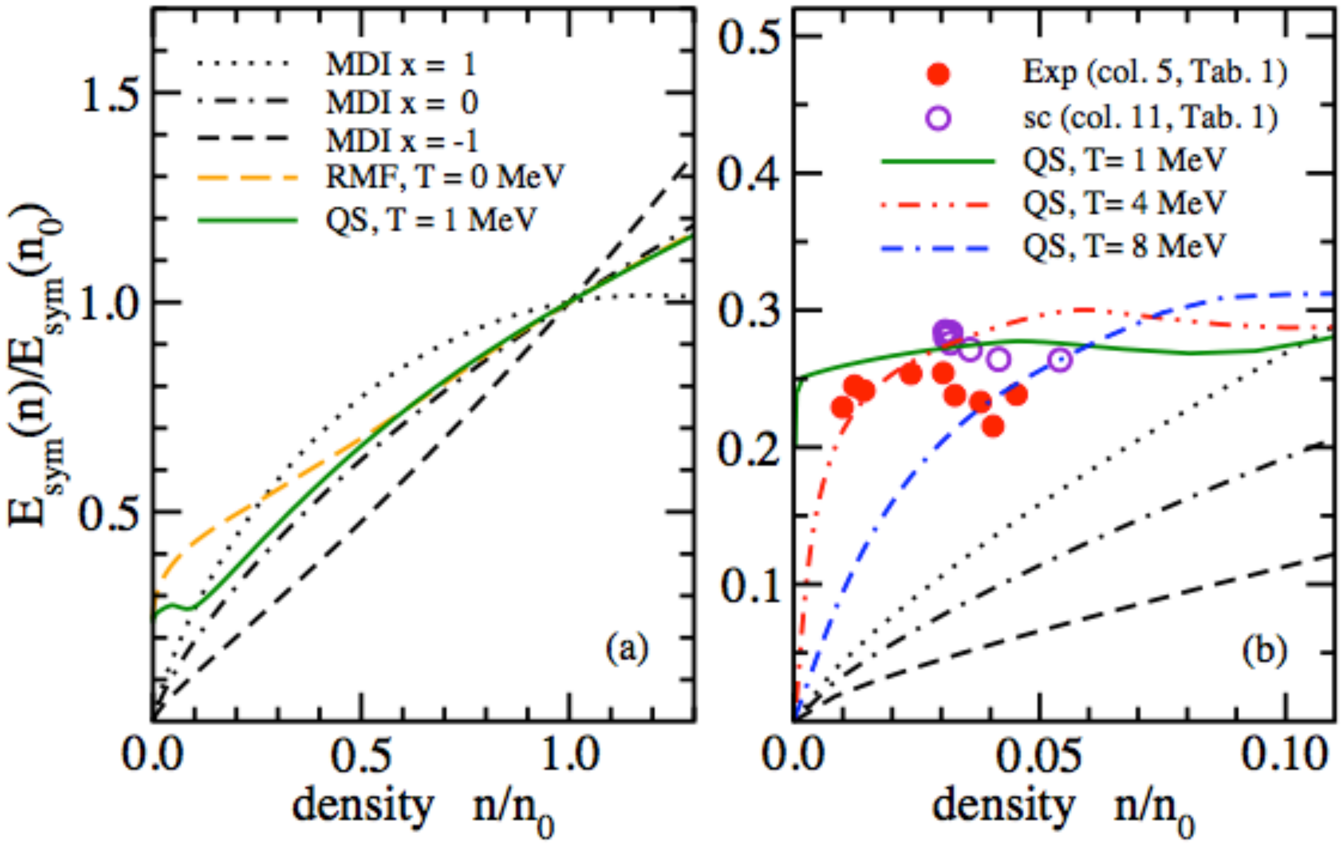
S. Kowalski et al., PRC **75**, 014601 (2007)

Symmetry energy, comparison experiment with theories



J.Natowitz et al., PRL 2010

Symmetry Energy

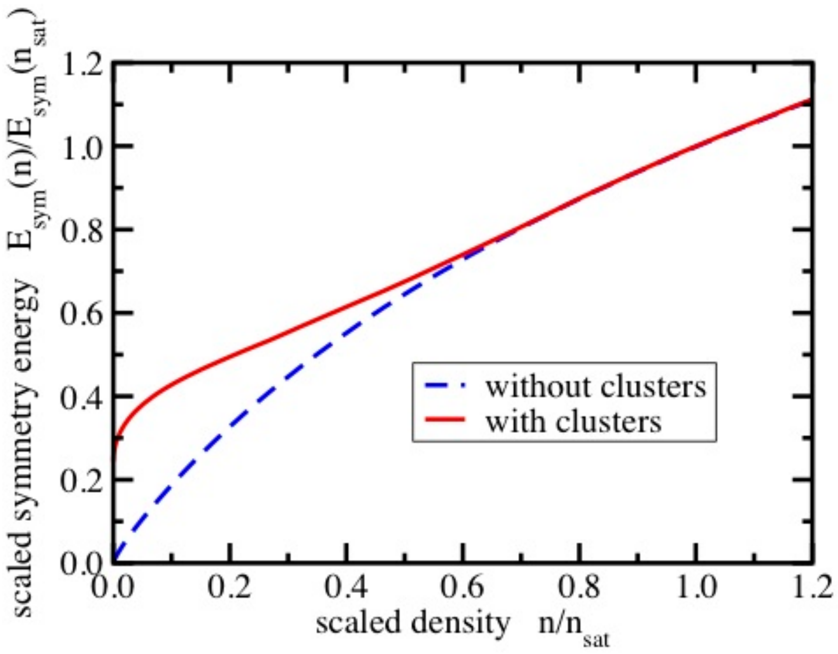


Scaled internal symmetry energy as a function of the scaled total density.
MDI: Chen et al., QS: quantum statistical, Exp: experiment at TAMU

J.Natowitz et al. PRL, May 2010

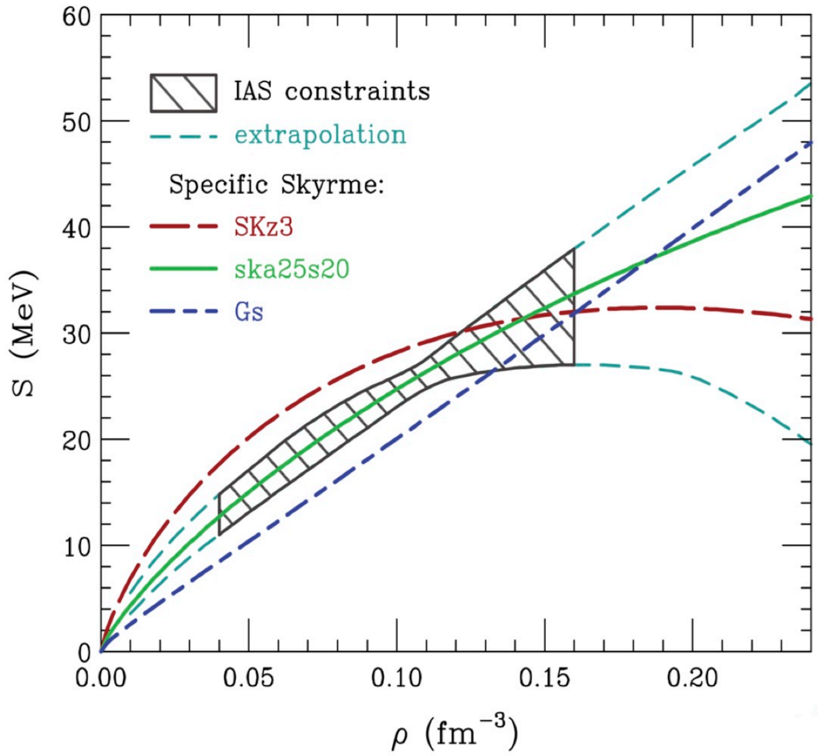
Symmetry energy, low density limit

Contribution of correlations/cluster



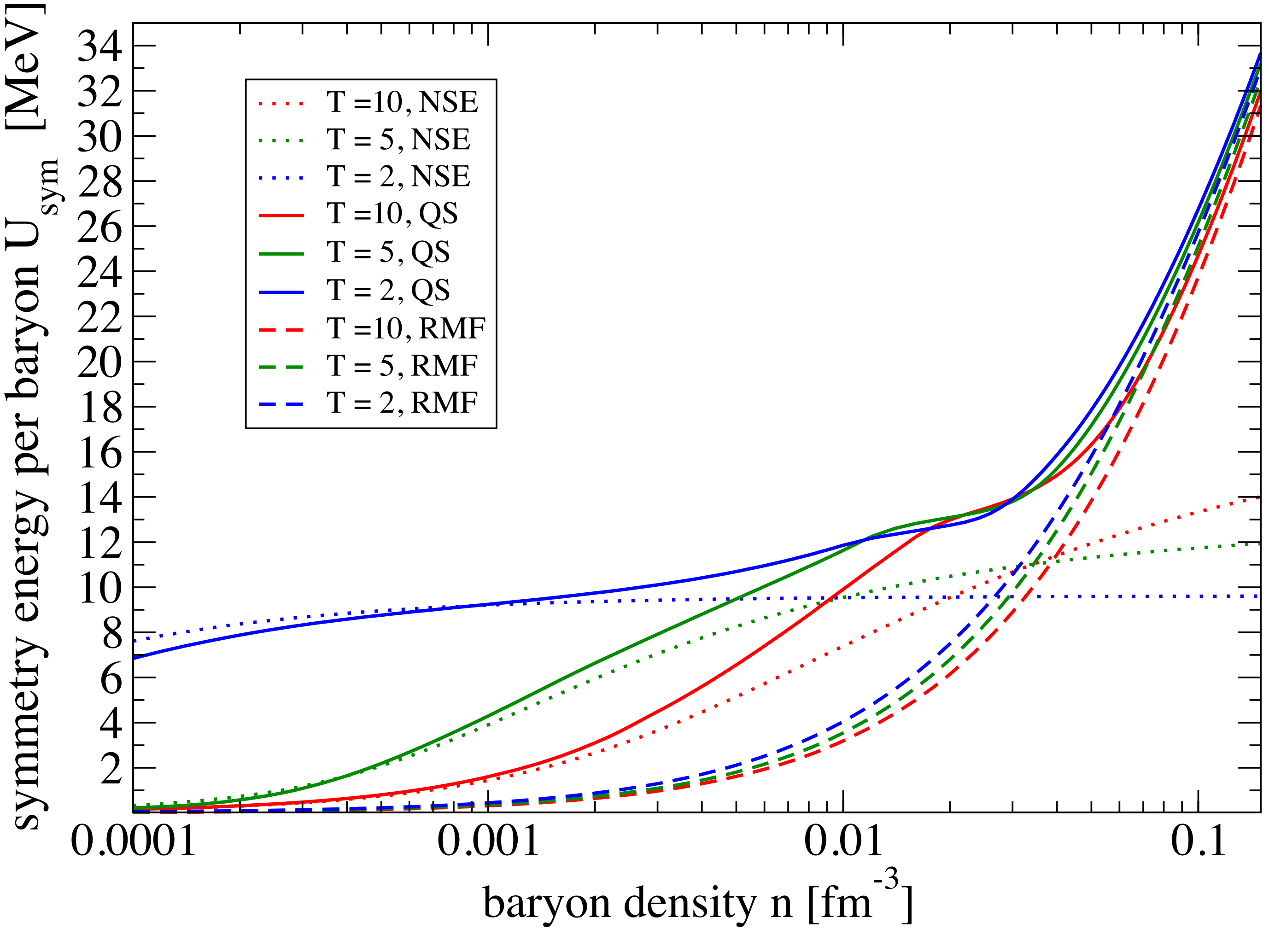
S. Typel, NN2012

(P. Danielewicz)

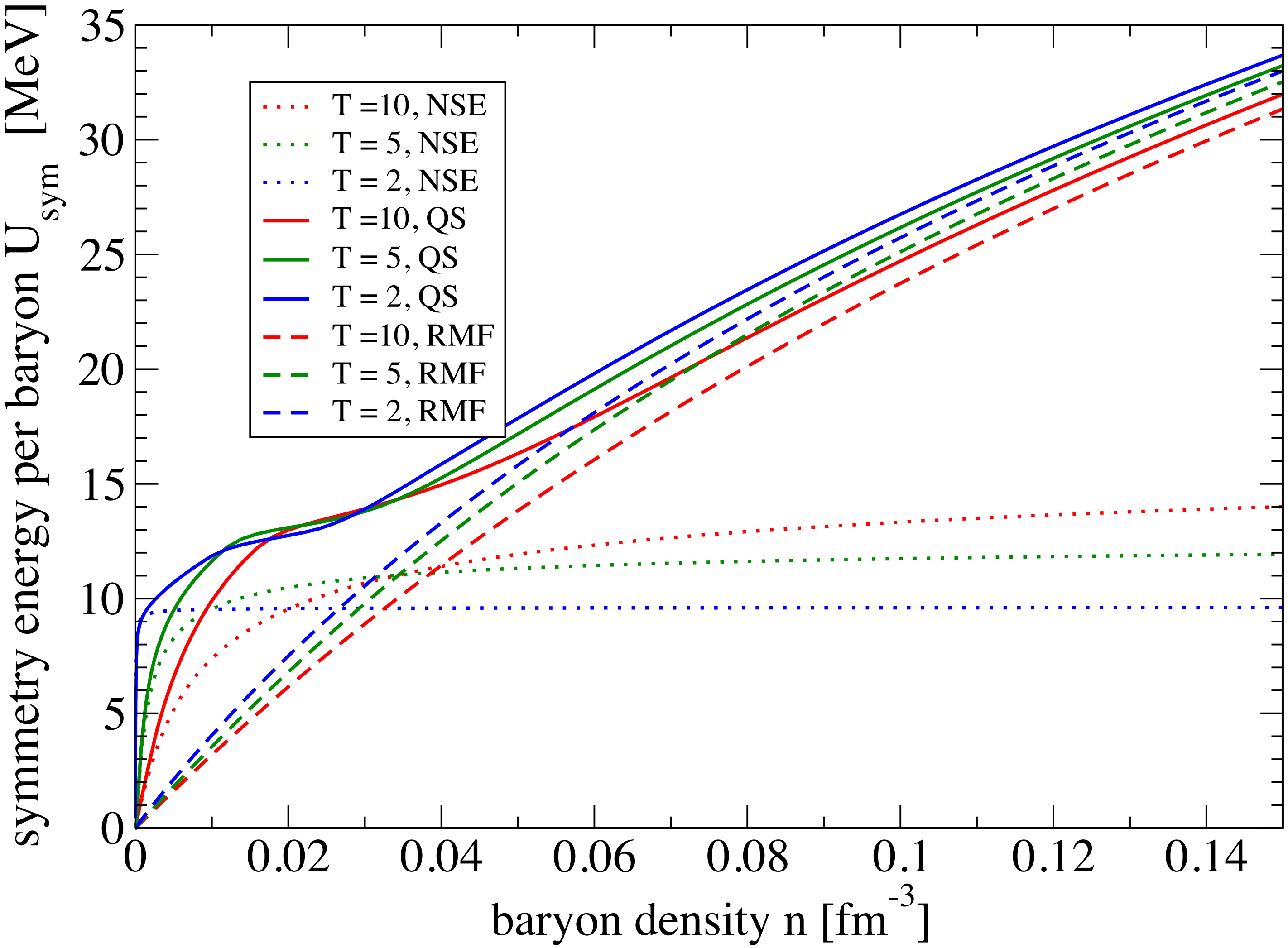


T. Fischer, Wroclaw 2012

Internal symmetry energy



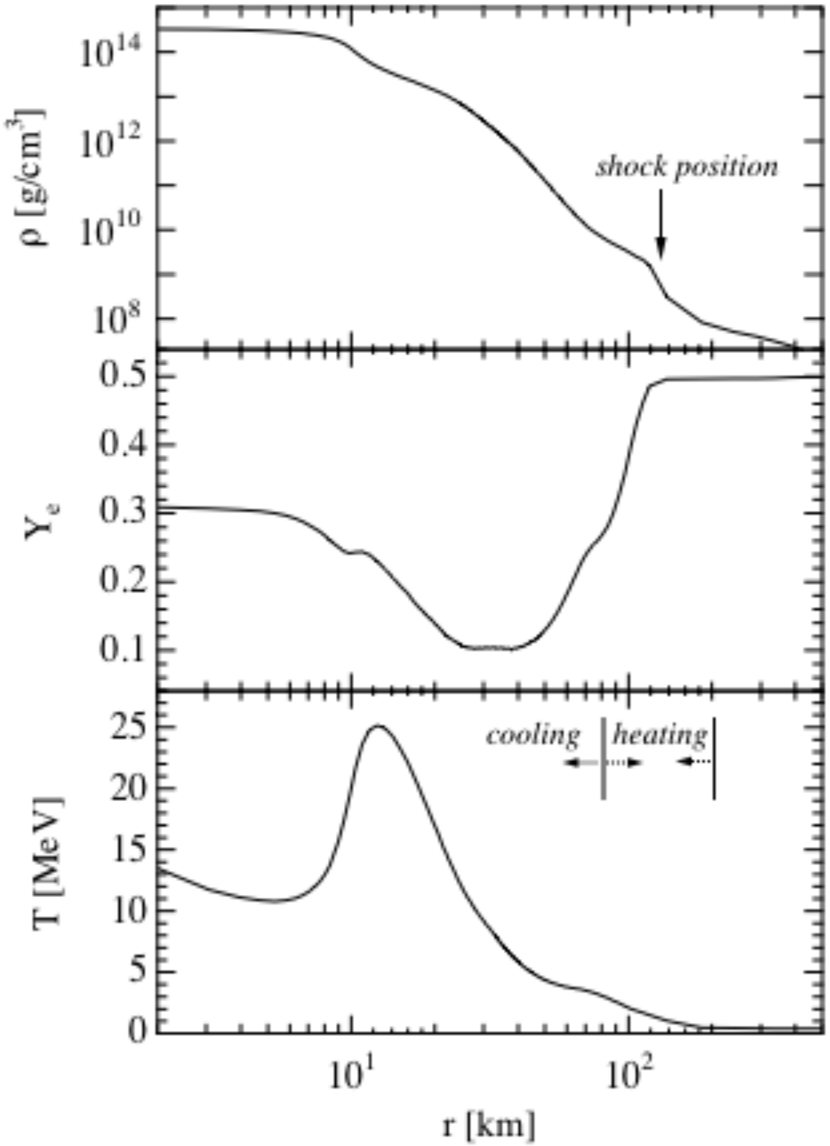
Internal symmetry energy



Future perspectives

- Astrophysics:
  heavy elements, pasta phases,…
- Nuclear structure:
  correlations (alpha-like clustering) in low density regions (Hoyle state, surfaces,…)
- Nonequilibrium processes:
  formation of correlations/ bound states in expanding dense matter

Core-collapse supernovae



Density.

electron fraction, and

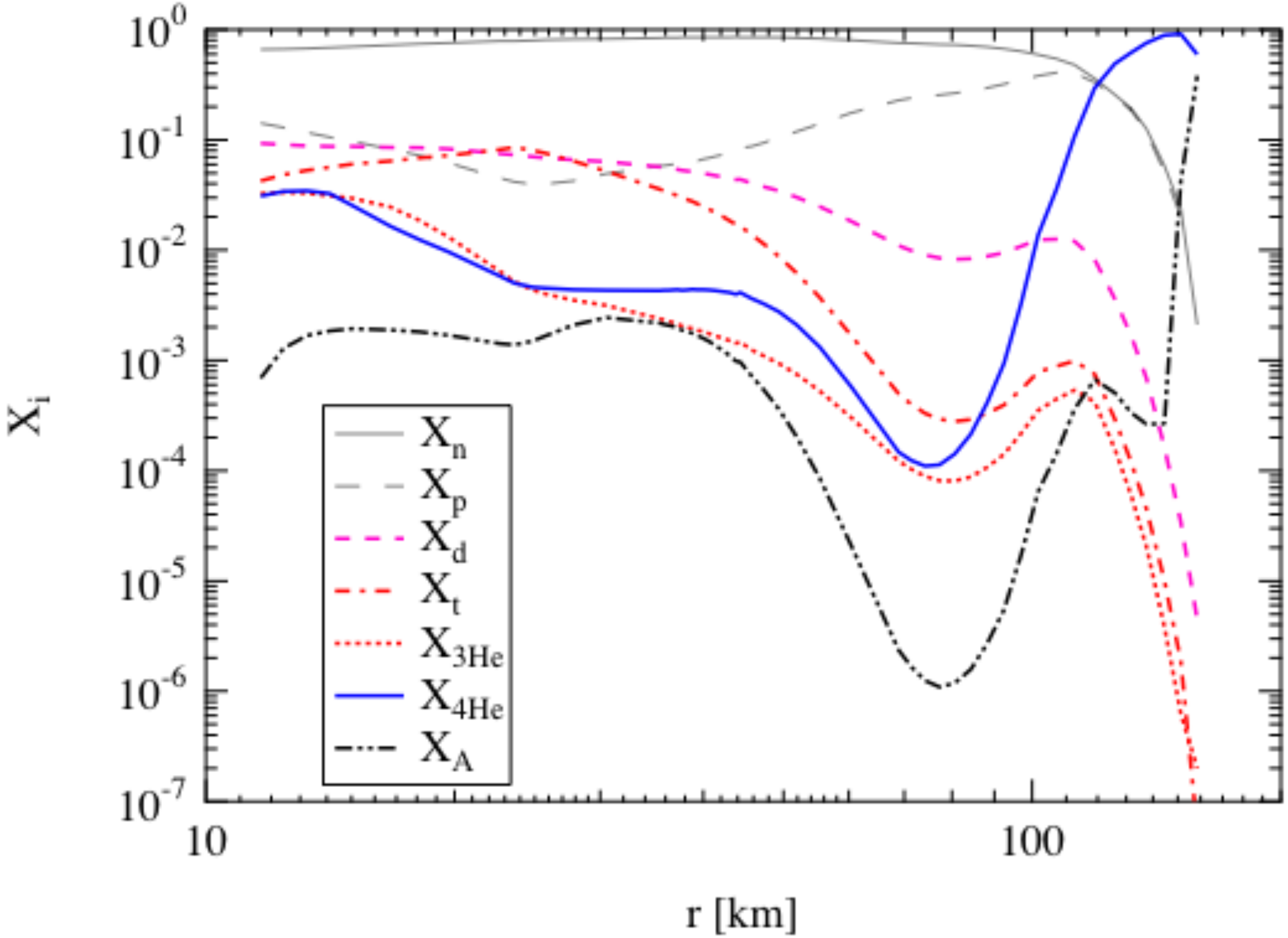
temperature profile

of a 15 solar mass supernova
at 150 ms after core bounce
as function of the radius.

Influence of cluster formation
on neutrino emission
in the cooling region and
on neutrino absorption
in the heating region ?

K.Sumiyoshi et al.,
Astrophys.J. **629**, 922 (2005)

Composition of supernova core



K.Sumiyoshi, G. R., PRC **77**, 055804 (2008)

Mass fraction X of light clusters for a post-bounce supernova core

Clusters in nuclei

- Low-density isomers (^{8}Be, ^{12}C, ^{16}O, …)

- Condensate wave function

- Suppression of the condensate with increasing density
- Dissolution of clusters with increasing density

Alpha gas-like states:
near threshold low density excited states
(Hoyle state, ^{16}O, ^{20}Ne,…)

Bo Zhou, Y. Funaki, H. Horiuchi, Zhongzhou Ren, G. Ropke, P. Schuck, A. Tohsaki, Chang Xu, and T. Yamada, arXiv:1304.1244 accepted PRL

Excited light nuclei

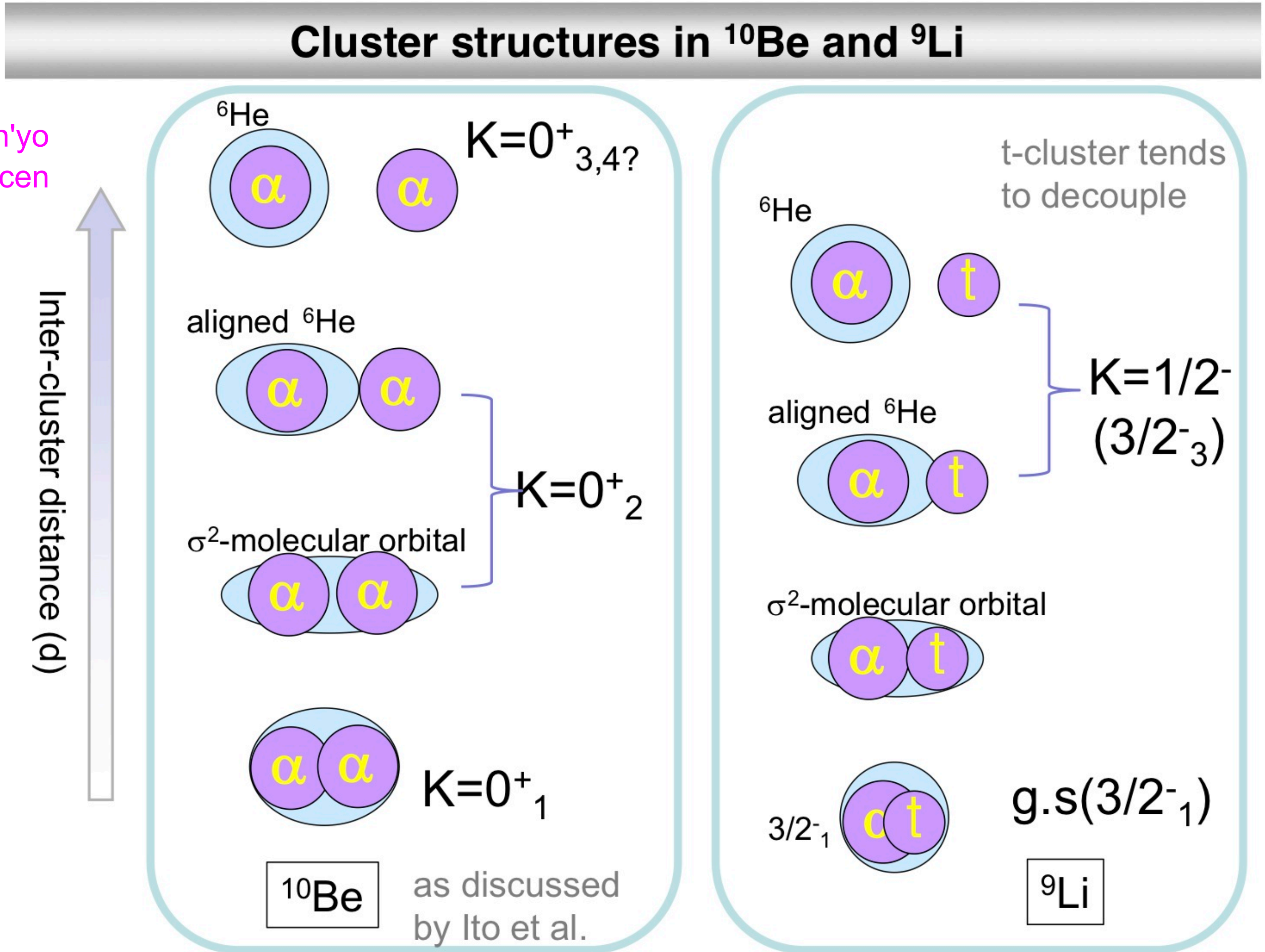
Yoshiko Kanada-En'yo
Cluster2012,Debrecen

decreasing density

deuterons?

systematics in weakly bound light elements

light clusters in neutron matter



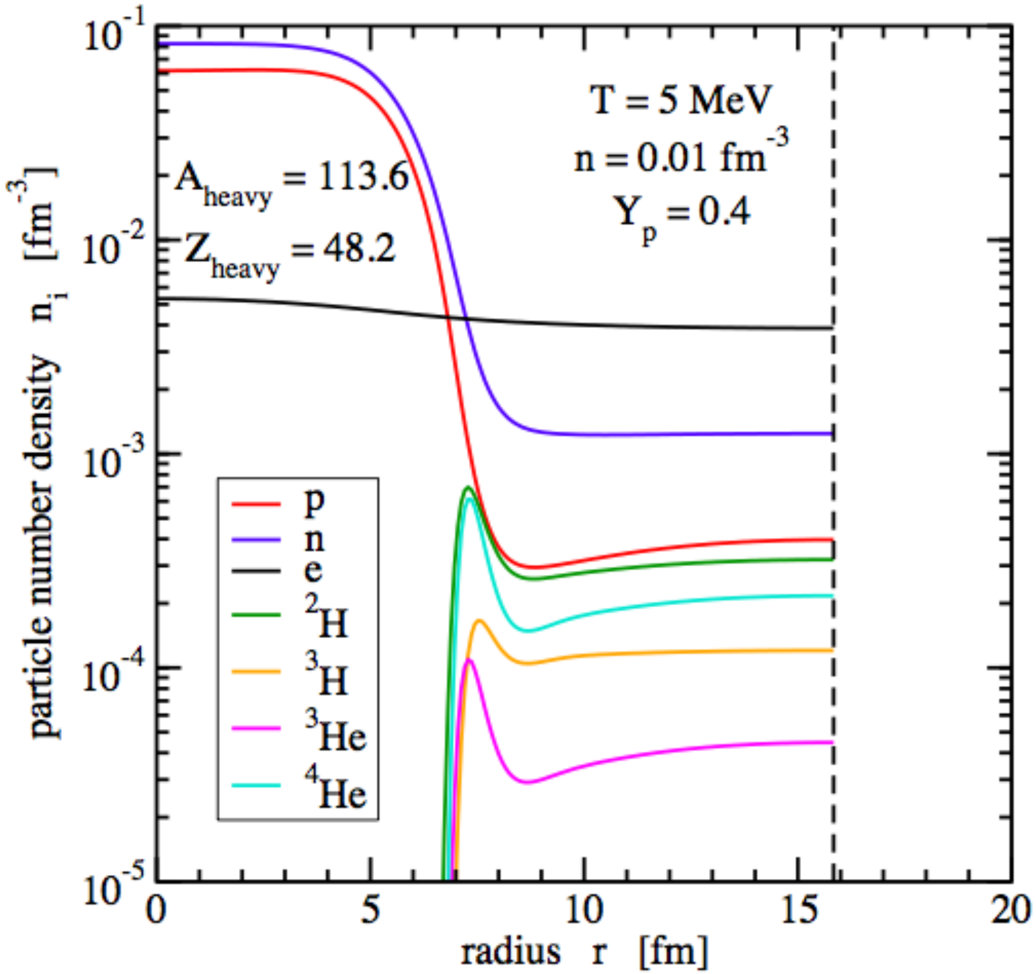
α cluster in astrophysics

Crust of neutron stars

Protons in droplets
(heavy nuclei)

α−cluster outside,
at the surface,
condensate?

S. Typel, Proc. Int. Workshop XII Hadron Physics



Chemical freeze-out

- Nonequilibrium, expanding source,
  Zubarev approach, transport model calculations
- Phase transition, instabilities, spinodal decomposition
- Correlated matter, QS approach,
  in-medium effects

Thermal analysis of particle yields from AGS to RHIC energies

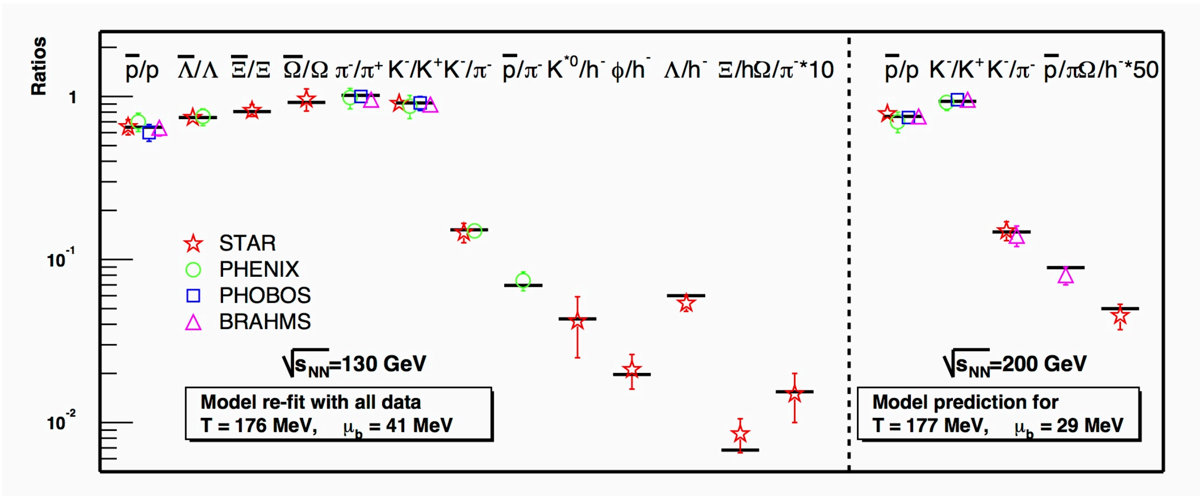


Fig. 8. Comparison of the experimental data on different particle multiplicity ratios obtained at RHIC at $\sqrt{s_{NN}} = 130$ and 200 GeV with thermal model calculations. The thermal model analysis is from Refs. (35, 36) and recent update by D. Magestro.

P. Braun-Munzinger, K. Redlich, J. Stachel, Particle Production in Heavy Ion Collisions arxuv: 0304013

Second summary

Verification from Heavy Ion Collisions

- Test of the nuclear matter equation of state:
  in-medium corrections are necessary
- Symmetry energy of nuclear matter: clustering effects at low densities must be taken into account

Future perspectives

- Astrophysics:
  heavy elements, pasta phases,…
- Nuclear structure:
  correlations (alpha-like clustering) in low density regions
  (Hoyle state, surfaces,…)
- Nonequilibrium processes:
  formation of correlations/ bound states in expanding dense matter

Thanks

to D. Blaschke, C. Fuchs, Y. Funaki, H. Horiuchi,
J. Natowitz, T. Klaehn, S. Shlomo, P. Schuck,
A. Sedrakian, K. Sumiyoshi, A. Tohsaki, S. Typel, H. Wolter, T. Yamada
for collaboration

to you
for attention

To Joe

Thanks for longstanding and intense collaboration.
Hope to continue.

I appreciate your interest in many-particle phenomena,
good intuition in physics, lively discussions and experience,
your nice personality and friendship.

QS versus NSE: comparison with data

Yields of p, (n), d, t, ^{3}He, ^{4}He

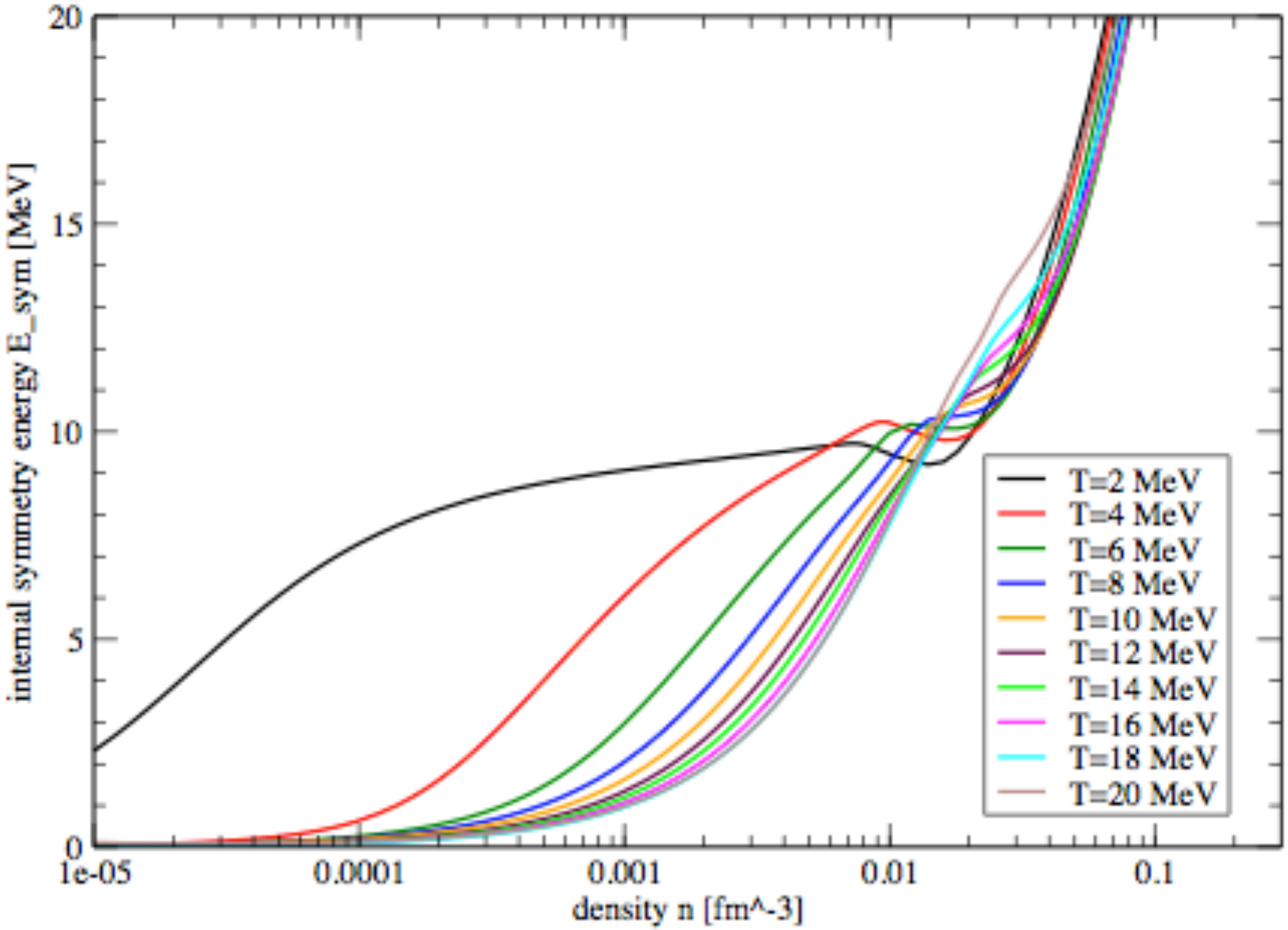
$$\epsilon = \frac{E_\alpha + E_d - E_t - E_h}{2E_\alpha - E_t - 2E_h}$$

$$4^\epsilon \left(\frac{27}{16}\right)^{3\epsilon/2} \frac{3}{4} \left(\frac{9}{8}\right)^{3/2} \frac{Y_\alpha^{2\epsilon-1} Y_p^\epsilon}{Y_h^{2\epsilon-1} Y_t^{\epsilon-1} Y_d}$$

Values:
NSE: 1
Yanello: 1.1
Kowalski: 1.23
Lijun: 1.36

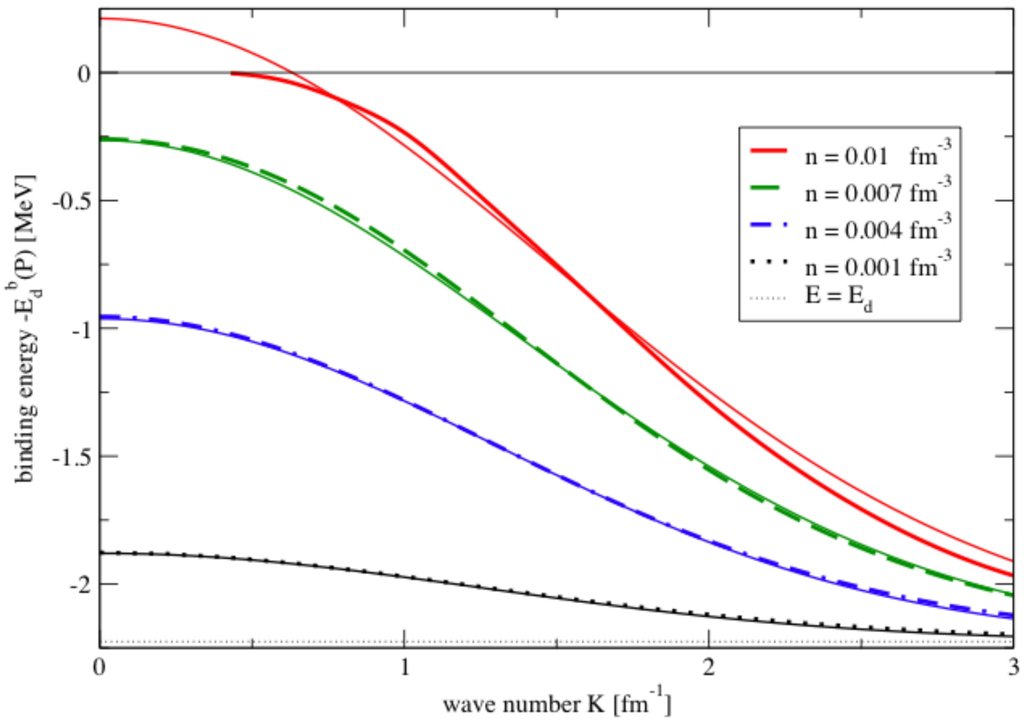
Discussion:
Nonequilibrium
Source
Medium effects

Internal symmetry energy



Shift of the deuteron binding energy

Dependence on center of mass momentum,
various densities, T=10 MeV



thin lines:

fit formula

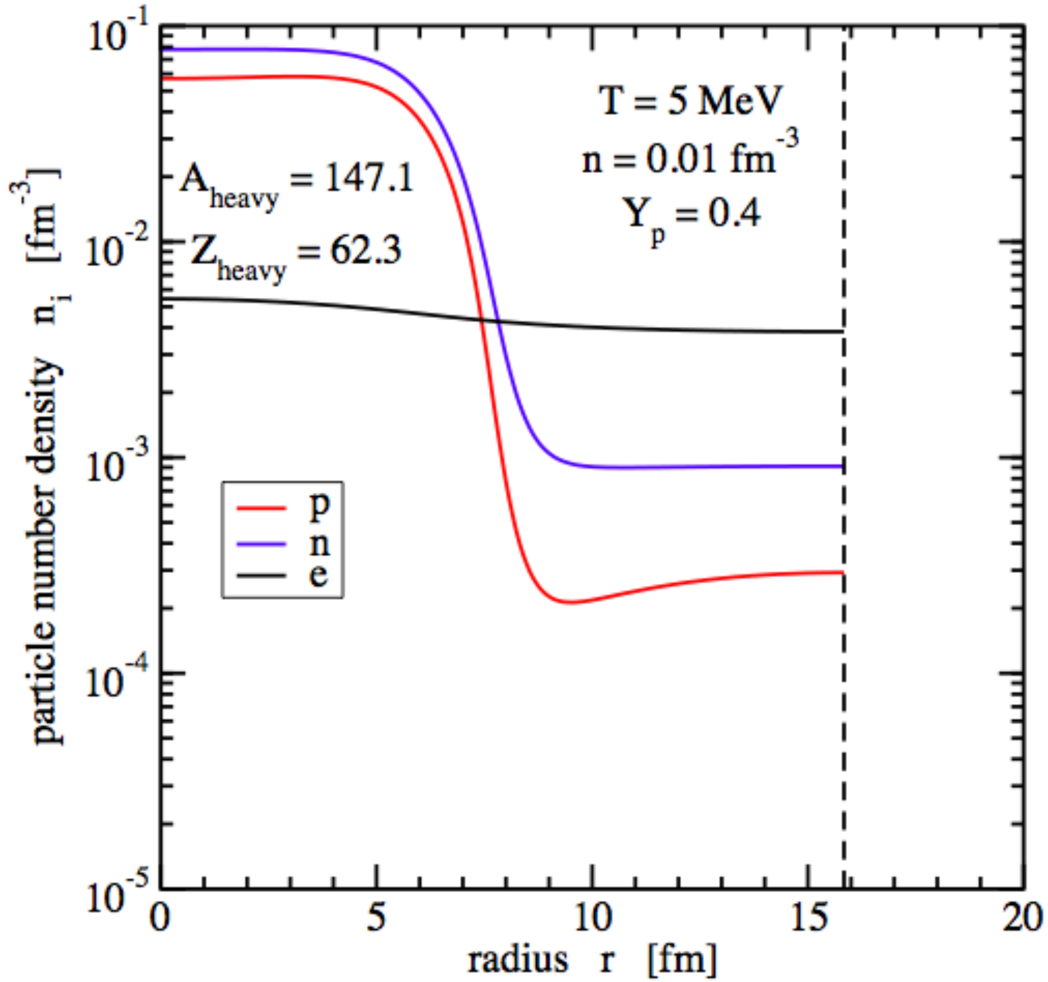
G.R., NP A **867**, 66 (2011)

α cluster in astrophysics

Crust of neutron stars

Protons in droplets
(heavy nuclei)

α−cluster outside,
at the surface,
condensate?



S. Typel, Proc. Int. Workshop XII Hadron Physics

Many-particle theory

- Dyson equation and self energy (homogeneous system)

$$G(1, iz_\nu) = \frac{1}{iz_\nu - E(1) - \Sigma(1, iz_\nu)}$$

- Evaluation of $\Sigma(1, iz_\nu)$:
  perturbation expansion, diagram representation

$$A(1, \omega) = \frac{2\mathrm{Im}\,\Sigma(1, \omega + i0)}{[\omega - E(1) - \mathrm{Re}\,\Sigma(1, \omega)]^2 + [\mathrm{Im}\,\Sigma(1, \omega + i0)]^2}$$

approximation for self energy $\longrightarrow$ approximation for equilibrium correlation functions

alternatively: simulations, path integral methods

Different approximations

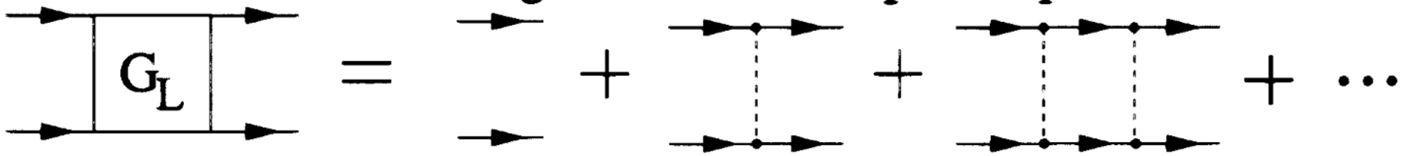
- **Expansion for small Im $\Sigma(1, \omega + i\eta)$**

$$A(1,\omega) \approx \frac{2\pi\delta(\omega - E^{\mathrm{quasi}}(1))}{1 - \frac{\mathrm{d}}{\mathrm{d}z}\mathrm{Re}\;\Sigma(1,z)|_{z=E^{\mathrm{quasi}}-\mu_1}}$$
$$-2\mathrm{Im}\;\Sigma(1,\omega+i\eta)\frac{\mathrm{d}}{\mathrm{d}\omega}\frac{P}{\omega+\mu_1-E^{\mathrm{quasi}}(1)}$$

quasiparticle energy $E^{\mathrm{quasi}}(1) = E(1) + \mathrm{Re}\;\Sigma(1,\omega)|_{\omega=E^{\mathrm{quasi}}}$

- **chemical picture: bound states $\hat{=}$ new species**

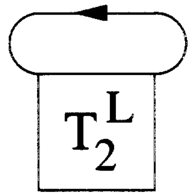
summation of ladder diagrams, Bethe-Salpeter equation



Different approximations

low density limit:

$$G_2^L(12, 1'2', i\lambda) = \sum_{n\mathbf{P}} \Psi_{n\mathbf{P}}(12) \frac{1}{i\omega_\lambda - E_{n\mathbf{P}}} \Psi^*_{n\mathbf{P}}(12)$$

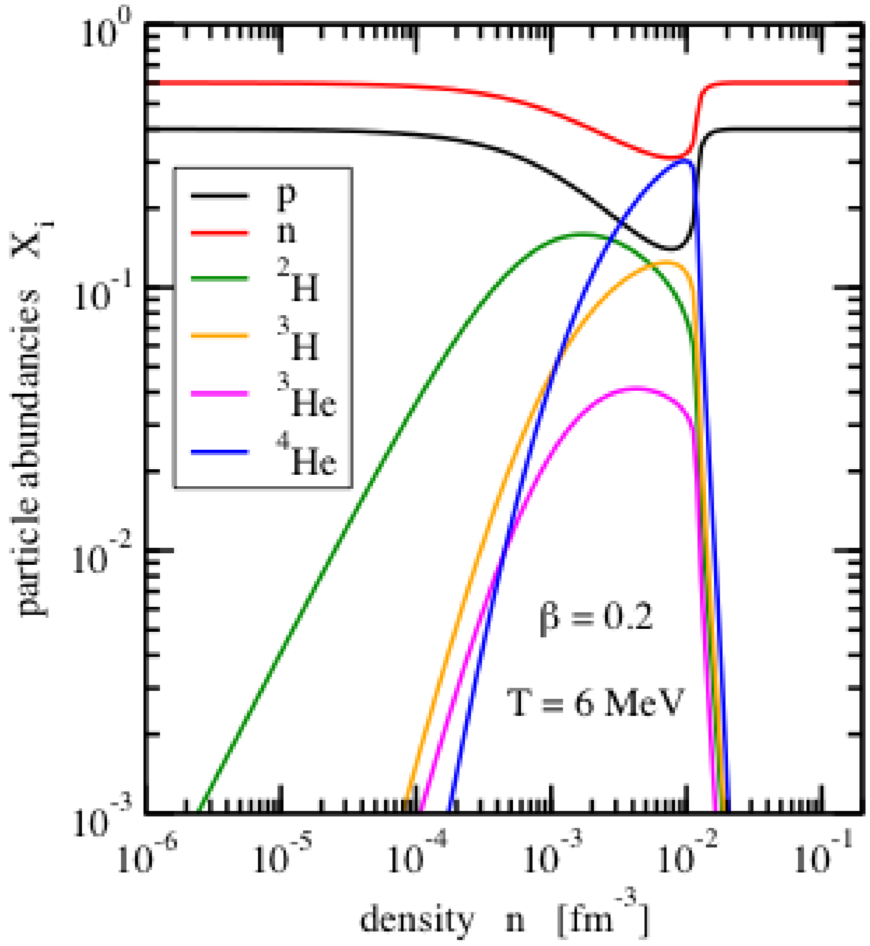
$\Sigma =$ 

$$n(\beta, \mu) = \sum_1 f_1(E^{\text{quasi}}(1)) + \sum_{2,n\mathbf{P}}^{\text{bound}} g_{12}(E_{n\mathbf{P}})$$

$$+ \sum_{2,n\mathbf{P}} \int_0^\infty \mathrm{d}k \; \delta_{\mathbf{k},\mathbf{p}_1-\mathbf{p}_2} g_{12}(E^{\text{quasi}}(1) + E^{\text{quasi}}(2)) 2 \sin^2 \delta_n(k) \frac{1}{\pi} \frac{\mathrm{d}}{\mathrm{d}k} \delta_n(k)$$

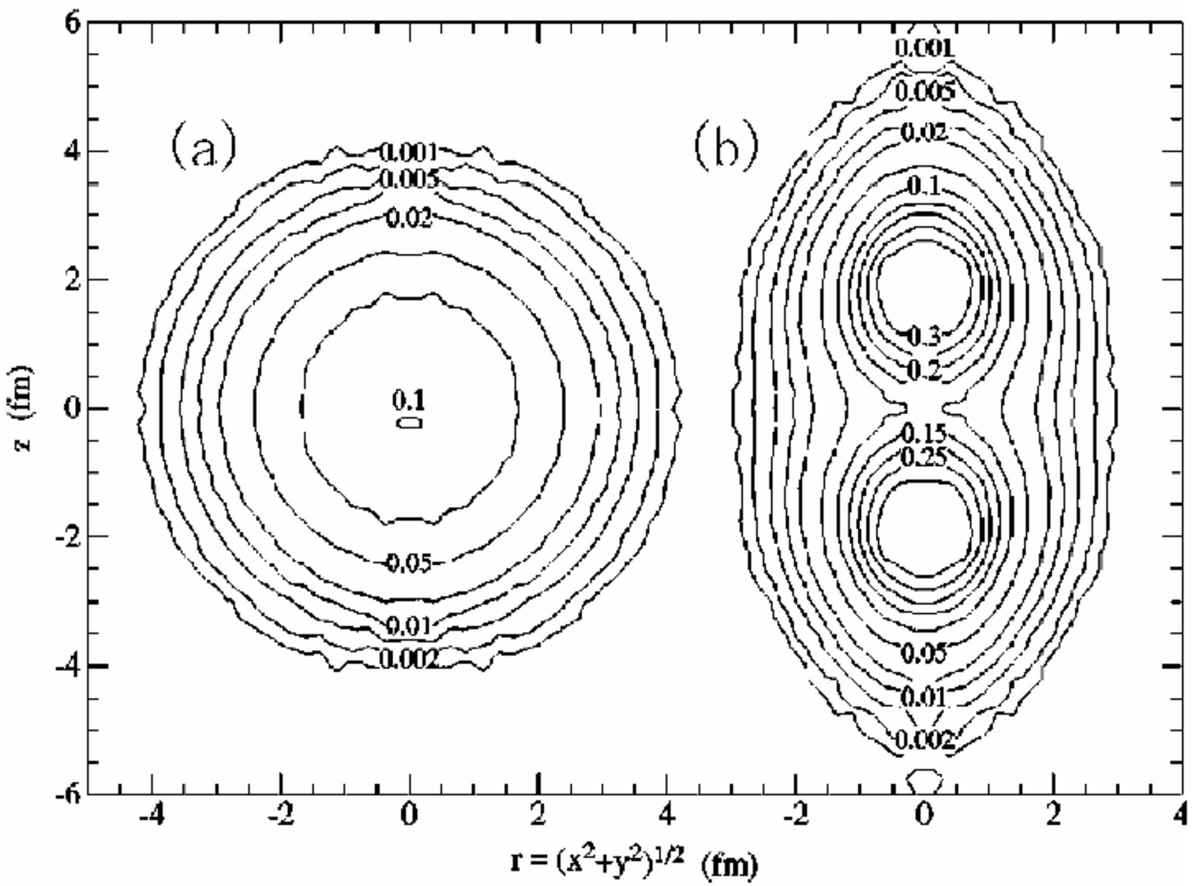
- generalized Beth-Uhlenbeck formula
  correct low density/low temperature limit:
  mixture of free particles and bound clusters

Light Cluster Abundances



S. Typel et al.,
PRC **81,** 015803 (2010)

α cluster structure of ^{8}Be



R.B. Wiringa et al., PRC **63**, 034605 (01)

Contours of constant density, plotted in cylindrical coordinates, for ^{8}Be(0+) .
The left side is in the laboratory frame while the right side is in the intrinsic frame.

Beyond shell model calculations

Correlated states cannot be described within the shell model alone in a simple way
Examples: - Hoyle like states
- preformed α particle in the nuclear surface of α−decaying heavy nuclei

mean field ansatz for the α-particle projected on good total momentum

$$\Psi_{1234} \propto \delta(\mathbf{K} - \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)\varphi(\mathbf{k}_1)\varphi(\mathbf{k}_2)\varphi(\mathbf{k}_3)\varphi(\mathbf{k}_4)$$

In general for the separation of the center-of-mass motion

$$\Psi_{1234} \propto \Phi(\mathbf{R})\psi_{int.}(|\mathbf{r}_i - \mathbf{r}_j|)$$

We may view ^{212}Po as an α-particle on top of a finite Fermi gas.
The lead core ^{208}Pb is doubly magic, one can treat it in mean field (Slater determinant).

However, as we know, adding more α's to the Pb core leads to deformed nuclei and the treatment of α's in the surface becomes a much more delicate subject.

Densitometers

- Albergo (NSE) cannot be used.
- Modified Albergo: use only the bound state yields, not the free nucleon yields [S. Kowalski et al., PRC **75**, 014601 (2007)]
- Natowitz: chemical constants $K_c(A,Z)$

  Volume is needed. Mekjian coalescence model.

  Good agreement with QS calculations
- Fluctuations?

- Only measured quantities Y_p, Y_d, Y_t, Y_h, Y_α ?
- Example:

$$K_\alpha = \frac{Y_\alpha}{Y_p^4} \frac{Y_h^2}{Y_t^2} (Y_p + Y_n + 2Y_d + 3Y_t + 3Y_h + 4Y_\alpha)^3$$

Low-density limit: NSE

$$\ln K_\alpha^{\mathrm{NSE}} = 3 \ln n_B + \frac{E_\alpha + 2E_h - 2E_t}{T} + \frac{9}{2} \ln \left(\frac{2\pi\hbar^2}{mT} \right) - \ln 2$$